

# REGULARITY OF VISCOSITY SOLUTIONS DEFINED BY HOPF-TYPE FORMULA FOR HAMILTON-JACOBI EQUATIONS

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**ABSTRACT.** Some properties of characteristic curves in connection with viscosity solutions of Hamilton-Jacobi equations defined by Hopf-type formula are studied. We investigate the points where the Hopf-type formula  $u(t, x)$  is differentiable, and the strip of the form  $(0, t_0) \times \mathbb{R}^n$  of the domain  $\Omega$  where the viscosity solution  $u(t, x)$  is continuously differentiable. Moreover, we present the propagation of singularity in forward of  $u(t, x)$ .

## 1. INTRODUCTION

The notion of viscosity solution introduced by Crandall M.G. and Lions P.L. [6] plays a fundamental role in studying Hamilton-Jacobi equations as well as the related problems such as calculus of variation, optimal control, etc. By definitions, a viscosity solution of Hamilton-Jacobi equation is merely a continuous function  $u$  satisfying differential inequalities or  $u$  is verified a such solution by  $C^k$ -test functions. As a result, the relationship between viscosity solutions and classical solutions is a subtle matter. Therefore, many authors pay attention to studying the regularity of viscosity solution in following meanings: Under what conditions that the viscosity solution  $u$  is locally Lipschitz or differentiable (may be almost everywhere in the domain of definition  $\Omega$  of  $u$ ); finding subregion  $V \subset \Omega$  where  $u \in C^1(V)$ ; investigating behaviour of sets where  $u$  is not differentiable, and so on. Most of these studies are based on the representation formulas of solutions where Hopf-Lax-Oleinik and Hopf formulas are especially concerned.

Consider the Cauchy problem for Hamilton-Jacobi equations of the form

$$(1.1) \quad \frac{\partial u}{\partial t} + H(t, x, D_x u) = 0, \quad (t, x) \in \Omega = (0, T) \times \mathbb{R}^n,$$

$$(1.2) \quad u(0, x) = \sigma(x), \quad x \in \mathbb{R}^n.$$

If the Hamiltonian  $H(t, x, p)$  is convex in  $p$ , the problem (1.1)-(1.2) is investigated via variational problem, and the representation of viscosity solutions of Hamilton-Jacobi equation as the value function associated to the problem may be considered a generalized form of Hopf-Lax-Oleinik formula

$$(1.3) \quad u(t, x) = \min_{y \in \mathbb{R}^n} \left\{ \sigma(y) + tH^*\left(\frac{x - y}{t}\right) \right\},$$

where  $H = H(p)$  is convex and superlinear,  $\sigma$  is Lipschitz on  $\mathbb{R}^n$ . Many results on the regularity of viscosity solutions in the case of convex Hamiltonians are obtained, see [1, 2, 4] especially [5] and references therein.

If  $H$  is nonconvex, Hopf formula for viscosity solution of the problem (1.1)-(1.2) is

$$(1.4) \quad u(t, x) = \max_{q \in \mathbb{R}^n} \{ \langle x, q \rangle - \sigma^*(q) - tH(q) \}$$

under the assumptions that  $H(t, x, p) = H(p)$  is a continuous function,  $\sigma(x)$  is convex and Lipschitz, see [3]. A generalization of formula (1.4) is that

$$(1.5) \quad u(t, x) = \max_{q \in \mathbb{R}^n} \{ \langle x, q \rangle - \sigma^*(q) - \int_0^t H(\tau, q) d\tau \}$$

where  $H = H(t, p)$  is a continuous and  $\sigma$  is convex, is a locally Lipschitz continuous function satisfying the initial condition and equation (1.1) at almost all points in the domain  $\Omega$ , i.e., a *Lipschitz solution*, but it is not a viscosity solution in general, see [8]. Recently, in [10] we prove that (1.5) defines a viscosity solution of the problem for a large class of Hamiltonians  $H = H(t, p)$ .

In this paper we study properties of characteristics of the Cauchy problem where  $H = H(t, p)$  in connection with formula (1.5). Then we present some results on the existence of strip of differentiability of the solution  $u(t, x)$  given by this formula as well as the points at which  $u(t, x)$  is not differentiable.

The structure of the paper is as follows. In section 2 we suggest a classification of characteristic curves at one point of the domain and then study the differentiability properties of Hopf-type formula  $u(t, x)$  on these curves. In section 3, we present the conditions related to characteristics so that  $u(t, x)$  defined by (1.5) is continuously differentiable on the strip  $(0, t_0) \times \mathbb{R}^n$ . Then we show that the singularities of solution  $u(t, x)$  may propagate forward from  $t$ -time  $t_0$  to the boundary of the domain.

This paper can be considered as a continuation of [9] to the case the dimension of state variable  $n$  is greater than 1. The results obtained here are new, even for Hamiltonian is independent of  $t$ . Our method here is to exploit the relationship between Hopf-type formula and characteristics based on the set of maximizers.

We use the following notations. Let  $T$  be a positive number,  $\Omega = (0, T) \times \mathbb{R}^n$ ;  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$  be the Euclidean norm and the scalar product in  $\mathbb{R}^n$ , respectively, and let  $B'(x_0, r)$  be the closed ball centered at  $x_0$  with radius  $r$ .

## 2. THE DIFFERENTIABILITY OF HOPF-TYPE FORMULA AND CHARACTERISTICS

We now consider the Cauchy problem for Hamilton-Jacobi equation:

$$(2.1) \quad \frac{\partial u}{\partial t} + H(t, D_x u) = 0, \quad (t, x) \in \Omega = (0, T) \times \mathbb{R}^n,$$

$$(2.2) \quad u(0, x) = \sigma(x), \quad x \in \mathbb{R}^n,$$

where the Hamiltonian  $H(t, p)$  is of class  $C([0, T] \times \mathbb{R}^n)$  and  $\sigma(x) \in C(\mathbb{R}^n)$  is a convex function.

Lets  $\sigma^*$  be the Fenchel conjugate of  $\sigma$ . We denote by  $D = \text{dom } \sigma^* = \{y \in \mathbb{R}^n \mid \sigma^*(y) < +\infty\}$  the effective domain of the convex function  $\sigma^*$ .

We assume a compatible condition for  $H(t, p)$  and  $\sigma(x)$  as follows.

(A1) : For every  $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$ , there exist positive constants  $r$  and  $N$  such that

$$\langle x, p \rangle - \sigma^*(p) - \int_0^t H(\tau, p) d\tau < \max_{|q| \leq N} \{ \langle x, q \rangle - \sigma^*(q) - \int_0^t H(\tau, q) d\tau \},$$

whenever  $(t, x) \in [0, T) \times \mathbb{R}^n$ ,  $|t - t_0| + |x - x_0| < r$  and  $|p| > N$ .

From now on, we denote by

$$(2.3) \quad u(t, x) = \max_{q \in \mathbb{R}^n} \{ \langle x, q \rangle - \sigma^*(q) - \int_0^t H(\tau, q) d\tau \}.$$

and

$$(2.4) \quad \varphi(t, x, q) = \langle x, q \rangle - \sigma^*(q) - \int_0^t H(\tau, q) d\tau, \quad (t, x) \in \Omega, \quad q \in \mathbb{R}^n.$$

For each  $(t, x) \in \Omega$ , let  $\ell(t, x)$  be the set of all  $p \in \mathbb{R}^n$  at which the maximum of the function  $\varphi(t, x, \cdot)$  is attained. In virtue of (A1),  $\ell(t, x) \neq \emptyset$ .

*Remark.* Condition (A1) is clearly satisfied if  $\sigma(x)$  is convex and Lipschitz on  $\mathbb{R}^n$ . Thus it can be considered as a generalization of the hypotheses used earlier, see [7, 3].

We record here a theorem that is necessary for further presentation.

**Theorem 2.1.** [12] Assume (A1). Then the function  $u(t, x)$  defined by (2.3) is a locally Lipschitz function satisfying equation (2.1) a.e. in  $\Omega$  and  $u(0, x) = \sigma(x)$ ,  $x \in \mathbb{R}^n$ . Furthermore,  $u(t, x)$  is of class  $C^1(V)$  in some open  $V \subset \Omega$  if and only if for every  $(t, x) \in V$ ,  $\ell(t, x)$  is a singleton.

*Remark 2.2.* If  $\ell(t_0, x_0) = \{p\}$  is a singleton, then all partial derivatives of  $u(t, x)$  at  $(t_0, x_0)$  exist and  $u_x(t_0, x_0) = p$ ,  $u_t(t_0, x_0) = -H(t_0, p)$  see ([13], p. 112). Moreover, we have:

**Theorem 2.3.** Assume (A1). Let  $(t_0, x_0) \in \Omega$  such that  $\ell(t_0, x_0)$  is a singleton. Then the function  $u(t, x)$  defined by (2.3) is differentiable at  $(t_0, x_0)$ .

*Proof.* For  $(h, k) \in \mathbb{R} \times \mathbb{R}^n$  small enough, let

$$\alpha = \limsup_{(h, k) \rightarrow (0, 0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k \rangle}{\sqrt{h^2 + |k|^2}},$$

where  $p \in \ell(t_0, x_0)$ ,  $p_t = -H(t_0, p)$ .

Then there exists a sequence  $(h_m, k_m) \rightarrow 0$  such that  $\lim_{m \rightarrow \infty} \Phi_m = \alpha$ , where

$$\Phi_m = \frac{u(t_0 + h_m, x_0 + k_m) - u(t_0, x_0) - p_t h_m - \langle p, k_m \rangle}{\sqrt{h_m^2 + |k_m|^2}}.$$

For each  $m \in \mathbb{N}$ , we choose  $p_m \in \ell(t_0 + h_m, x_0 + k_m)$  then

$$\begin{aligned} \Phi_m &\leq \frac{\varphi(t_0 + h_m, x_0 + k_m, p_m) - \varphi(t_0, x_0, p_m) - p_t h_m - \langle p, k_m \rangle}{\sqrt{h_m^2 + |k_m|^2}} \\ &\leq \frac{-h_m(p_t + H(\tau_m, p_m)) - \langle p_m - p, k_m \rangle}{\sqrt{h_m^2 + |k_m|^2}}, \end{aligned}$$

for some  $\tau_m$  lying between  $t_0$  and  $t_0 + h_m$ ;  $\varphi(t, x, p)$  is given by (2.4).

Taking into account the assumption (A1), it is easy to see that, for  $(h_m, k_m)$  small enough, the sequence  $(p_m)_m$  is bounded, then we can choose a subsequence also denoted by  $(p_m)_m$  such that  $p_m \rightarrow p_0$  as  $m \rightarrow \infty$ . Since the set-valued mapping  $(t, x) \mapsto \ell(t, x)$  is upper semicontinuous [12], then  $p_0 \in \ell(t_0, x_0)$ , that is  $p_0 = p$ .

Now, letting  $m \rightarrow \infty$  we have

$$\alpha = \lim_{m \rightarrow \infty} \Phi_m \leq \lim_{m \rightarrow \infty} \frac{-h_m(p_t + H(\tau_m, p_m)) - \langle p_m - p, k_m \rangle}{\sqrt{h_m^2 + |k_m|^2}} = 0.$$

On the other hand, let

$$\beta = \liminf_{(h,k) \rightarrow (0,0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k \rangle}{\sqrt{h^2 + |k|^2}}.$$

We have, for  $p \in \ell(t_0, x_0)$

$$\begin{aligned} u(t_0 + h, x_0 + k) - u(t_0, x_0) &\geq \varphi(t_0 + h, x_0 + k, p) - \varphi(t_0, x_0, p) \\ &\geq -hH(\tau^*, p) + \langle p, k \rangle, \end{aligned}$$

where  $\tau^*$  lies between  $t_0$  and  $t_0 + h$ . Therefore

$$\beta \geq \liminf_{(h,k) \rightarrow (0,0)} \frac{-h(-p_t - H(\tau^*, p))}{\sqrt{h^2 + |k|^2}} = 0.$$

Thus,

$$\lim_{(h,k) \rightarrow (0,0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k \rangle}{\sqrt{h^2 + |k|^2}} = 0.$$

The theorem is then proved.  $\square$

**Definition 2.4.** We call the function  $u(t, x)$  given by (2.3) the *Hopf-type formula* for Problem (2.1)-(2.2). A point  $(t_0, x_0) \in \Omega$  is said *regular* for  $u(t, x)$  if it is differentiable at this point. Other point is said *singular* if at which,  $u(t, x)$  is not differentiable.

Consequently, by Theorem 2.3, we see that  $(t_0, x_0) \in \Omega$  is regular if and only if  $\ell(t_0, x_0)$  is a singleton.

Next, in this section we focus on the study the differentiability of function  $u(t, x)$  given by Hopf-type formula on the characteristics. To this aim, let us recall the Cauchy method of characteristics for Problem (2.1)-(2.2).

From now on, we suppose that  $H(t, p)$  and  $\sigma(x)$  are of class  $C^1$ .

The characteristic differential equations of Problem (2.1)-(2.2) is as follows

$$(2.5) \quad \dot{x} = H_p ; \quad \dot{v} = \langle H_p, p \rangle - H ; \quad \dot{p} = 0$$

with initial conditions

$$(2.6) \quad x(0) = y ; \quad v(0) = \sigma(y) ; \quad p(0) = \sigma_y(y) , \quad y \in \mathbb{R}^n.$$

Then a characteristic strip of the problem (2.1)-(2.2) (i.e., a solution of the system of differential equations (2.5) - (2.6)) is defined by

$$(2.7) \quad \begin{cases} x = x(t, y) = y + \int_0^t H_p(\tau, \sigma_y(y)) d\tau, \\ v = v(t, y) = \sigma(y) + \int_0^t \langle H_p(\tau, \sigma_y(y)), \sigma_y(y) \rangle d\tau \\ \quad - \int_0^t H(\tau, \sigma_y(y)) d\tau, \\ p = p(t, y) = \sigma_y(y). \end{cases}$$

The first component of solutions (2.7) is called the characteristic curve (briefly, characteristics) emanating from  $y$  i.e., the curve defined by

$$(2.8) \quad \mathcal{C} : x = x(t, y) = y + \int_0^t H_p(\tau, \sigma_y(y)) d\tau, \quad t \in [0, T].$$

Let  $(t_0, x_0) \in \Omega$ . Denoted by  $\ell^*(t_0, x_0)$  the set of all  $y \in \mathbb{R}^n$  such that there is a characteristic curve emanating from  $y$  and passing the point  $(t_0, x_0)$ . We have  $\ell(t_0, x_0) \subset \sigma_y(\ell^*(t_0, x_0))$ , see [9]. Therefore  $\ell^*(t_0, x_0) \neq \emptyset$ .

**Proposition 2.5.** *Let  $(t_0, x_0) \in \Omega$ . Then a characteristic curve passing  $(t_0, x_0)$  has form*

$$(2.9) \quad x = x(t, y) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y)) d\tau, \quad t \in [0, T]$$

for some  $y \in \ell^*(t_0, x_0)$ .

*Proof.* Let  $C : x = x(t, y) = y + \int_0^t H_p(\tau, \sigma_y(y)) d\tau$  be a characteristic curve passing  $(t_0, x_0)$ . By definition,  $y \in \ell^*(t_0, x_0)$ . Then we have

$$x_0 = y + \int_0^{t_0} H_p(\tau, \sigma_y(y)) d\tau$$

Therefore,

$$x = x_0 - \int_0^{t_0} H_p(\tau, \sigma_y(y)) d\tau + \int_0^t H_p(\tau, \sigma_y(y)) d\tau = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y)) d\tau.$$

Conversely, let  $C_1 : x = x(t, y) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y)) d\tau$  for  $y \in \ell^*(t_0, x_0)$  be some curve passing  $(t_0, x_0)$ . Then we can rewrite  $C_1$  as:

$$(2.10) \quad x = x_0 - \int_0^{t_0} H_p(\tau, \sigma_y(y)) d\tau + \int_0^t H_p(\tau, \sigma_y(y)) d\tau.$$

On the other hand, let  $C_2$  :

$$(2.11) \quad x = y + \int_0^t H_p(\tau, \sigma_y(y)) d\tau$$

be the characteristic curve also passing  $(t_0, x_0)$ . Besides that, both  $C_1, C_2$  are integral curves of the ODE  $x' = H_p(t, \sigma_y(y))$ , thus they must coincide. This proves the proposition.  $\square$

*Remark 2.6.* Suppose that  $\sigma_y(y) = p_0 \in \ell(t_0, x_0)$  then  $y$  is in the subgradient of convex function  $\sigma^*$  at  $p_0 : y \in \partial\sigma^*(p_0)$ . Moreover, from (2.10) and (2.11), we have  $y = x_0 - \int_0^{t_0} H_p(\tau, p_0) d\tau$ .

Now, let  $\mathcal{C}$  be a characteristic curve passing  $(t_0, x_0)$  that is written as

$$x = x(t, y) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y)) d\tau$$

We say that the characteristic curve  $\mathcal{C}$  is of the *type (I)* at point  $(t_0, x_0) \in \Omega$ , if  $\sigma_y(y) = p \in \ell(t_0, x_0)$ . If  $\sigma_y(y) \in \sigma_y(\ell^*(t_0, x_0)) \setminus \ell(t_0, x_0)$  then  $\mathcal{C}$  is said of *type (II)* at  $(t_0, x_0)$ .

In the next, we assume an additional condition for the Hamiltonian  $H = H(t, p)$ .

(A2): The Hamiltonian  $H(t, p)$  is admitted as one of two following forms:

a)  $H(t, \cdot)$  is a convex or concave function for all  $t \in (0, T)$ .

b)  $H(t, p) = g(t)h(p) + k(t)$  for some functions  $g, h, k$  where  $g(t)$  does not change its sign for all  $t \in (0, T)$ .

*Remark 2.7.* 1. In particular, if  $H(t, p) = H(p)$  then the condition (A2), b) is obviously satisfied.

2. In [10] we proved that if assumptions (A1) and (A2) are satisfied, then the function  $u(t, x)$  defined by Hopf-type formula (2.3) is a viscosity solution of Problem (2.1)-(2.2). Moreover, if  $\sigma(x)$  is Lipschitz on  $\mathbb{R}^n$  then  $u(t, x)$  is a semiconvex function.

The following lemma is helpful in studying Fenchel conjugate of  $C^1$ -convex function.

**Lemma 2.8.** *Let  $v$  be a convex function and  $D = \text{dom } v \subset \mathbb{R}^n$ . Suppose that there exist  $p, p_0 \in D$ ,  $p \neq p_0$  and  $y \in \partial v(p_0)$  such that*

$$\langle y, p - p_0 \rangle = v(p) - v(p_0).$$

*Then for all  $z$  in the straight line segment  $[p, p_0]$  we have*

$$v(z) = \langle y, z \rangle - \langle y, p_0 \rangle + v(p_0).$$

*Moreover,  $y \in \partial v(z)$  for all  $z \in [p, p_0]$ .*

*Proof.* For  $z = \lambda p + (1 - \lambda)p_0 \in [p, p_0]$ ,  $\lambda \in [0, 1]$ , we have

$$v(z) \leq \lambda v(p) + (1 - \lambda)v(p_0) = \lambda(v(p) - v(p_0)) + v(p_0).$$

From the hypotheses, we have

$$\begin{aligned} v(z) &\leq \lambda \langle y, p - p_0 \rangle + v(p_0) \\ &\leq \langle y, \lambda p + (1 - \lambda)p_0 - p_0 \rangle + v(p_0). \end{aligned}$$

On the other hand, since  $y \in \partial v(p_0)$ , then

$$\langle y, \lambda p + (1 - \lambda)p_0 - p_0 \rangle \leq v(z) - v(p_0).$$

Thus

$$v(z) = \langle y, z \rangle - \langle y, p_0 \rangle + v(p_0).$$

Next, let  $z \in [p, p_0]$ . For any  $x \in D$ , we have

$$\begin{aligned} v(x) - v(z) &= v(x) - \langle y, z \rangle + \langle y, p_0 \rangle - v(p_0) \\ &= v(x) - v(p_0) - \langle y, z - p_0 \rangle \\ &\geq \langle x - p_0, y \rangle - \langle z - p_0, y \rangle \\ &\geq \langle x - z, y \rangle. \end{aligned}$$

This gives us that  $y \in \partial v(z)$ . □

Now we present properties of characteristic curves of type (I) at  $(t_0, x_0)$  given by the following theorems.

**Theorem 2.9.** *Assume (A1) and (A2). Let  $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ ,  $p_0 = \sigma_y(y) \in \ell(t_0, x_0)$  and let*

$$(2.12) \quad \mathcal{C} : x = x(t) = x_0 + \int_{t_0}^t H_p(\tau, p_0) d\tau, (t, x) \in \Omega$$

*be a characteristic curve of type (I) at  $(t_0, x_0)$ . Then  $p_0 \in \ell(t, x)$  and moreover,  $\ell(t, x) \subset \ell(t_0, x_0)$  for all  $(t, x) \in \mathcal{C}$ ,  $0 \leq t \leq t_0$ .*

*Proof.* Let  $(t_1, x_1) \in \mathcal{C}$ ,  $0 \leq t_1 \leq t_0$ . Take an arbitrary  $p \in \mathbb{R}^n$  and denote by

$$\eta(t, p) = \varphi(t, x, p) - \varphi(t, x, p_0), \quad (t, x) \in \mathcal{C}, \quad t \in [0, t_0],$$

where  $\varphi(t, x, p) = \langle x, p \rangle - \sigma^*(p) - \int_0^t H(\tau, p) d\tau$ . Then

$$(2.13) \quad \eta(t, p) = \langle x(t), p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) - \int_0^t (H(\tau, p) - H(\tau, p_0)) d\tau$$

for  $(t, x) \in \mathcal{C}$ .

We shall prove that  $\eta(t, p) \leq 0$  for all  $t \in [0, t_0]$ .

It is obviously that,  $\eta(t_0, p) \leq 0$ . On the other hand, from (2.13) and Remark 2.6, we have

$$\eta(0, p) = \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)),$$

where  $y \in \partial \sigma^*(p_0)$ . By a property of subgradient of convex function, we have

$$(2.14) \quad \eta(0, p) = \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) \leq 0.$$

As a result, we have  $\eta(0, p) \leq 0$ ;  $\eta(t_0, p) \leq 0$ .

Since  $x = x(t) = x_0 + \int_{t_0}^t H_p(\tau, p_0) d\tau$ , then from (2.13) we also have

$$\eta'(t, p) = \langle H_p(t, p_0), p - p_0 \rangle - (H(t, p) - H(t, p_0)), \quad t \in [0, t_0].$$

Consider the following cases:

**Case 1.** If  $H(t, \cdot)$  is convex, then

$$\langle H_p(t, p_0), p - p_0 \rangle \leq H(t, p) - H(t, p_0).$$

Therefore  $\eta'(t, p) \leq 0$ , for all  $t \in [0, t_0]$ .

Similarly, if  $H(t, \cdot)$  is a concave function, we have  $\eta'(t, p) \geq 0$ , for all  $t \in [0, t_0]$ .

**Case 2.** If  $H(t, p) = g(t)h(p) + k(t)$ , and  $g(t)$  does not change its sign in  $(0, T)$ , then

$$\begin{aligned} \eta'(t, p) &= \langle g(t)h_p(p_0), p - p_0 \rangle - g(t)(h(p) - h(p_0)) \\ &= (\langle h_p(p_0), p - p_0 \rangle - (h(p) - h(p_0)))g(t) = \lambda g(t), \end{aligned}$$

where  $\lambda = \langle h_p(p_0), p - p_0 \rangle - (h(p) - h(p_0))$  is a constant. Therefore,  $\eta'(t, p)$  also does not change its sign on  $[0, t_0]$ .

Combining the two cases above, we have, for all  $t \in [0, t_0]$  :

(i) If  $\eta'(t, p) \geq 0$  then  $\eta(t_1, p) \leq \eta(t_0, p) \leq 0$ .

(ii) If  $\eta'(t, p) \leq 0$  then  $\eta(t_1, p) \leq \eta(0, p) \leq 0$ .

Thus we obtain  $\varphi(t_1, x_1, p) \leq \varphi(t_1, x_1, p_0)$  for all  $p \in \mathbb{R}^n$ . Consequently,  $p_0 \in \ell(t_1, x_1)$  for any  $(t_1, x_1) \in \mathcal{C}$ ,  $t_1 \in [0, t_0]$ .

Now, let  $p \notin \ell(t_0, x_0)$ . Then, depending on  $\eta'(t, p) \geq 0$  or  $\eta'(t, p) \leq 0$ , we have

$$\eta(t, p) \leq \eta(t_0, p) < 0, \quad \text{or}$$

$$\eta(t, p) \leq \eta(0, p) = \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)), \quad t \in [0, t_0].$$

Since  $p \neq p_0$ , then  $\langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) < 0$ . Actually, if it is false, i.e.,  $\langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = 0$ , then applying Lemma 2.8, we see that  $[p, p_0]$  is contained in  $\mathcal{D} = \{z \in \text{dom}\sigma^* \mid \partial\sigma^*(z) \neq \emptyset\}$  and  $\sigma^*$  is not strictly convex on the set  $[p, p_0]$ . This is a contradiction, since  $\sigma(x)$  is of  $C^1(\mathbb{R}^n)$ , then it is essentially strictly convex on  $D$ . In particular,  $\sigma^*$  is strictly convex on  $[p, p_0]$ , see ([11], Thm. 26.3). Therefore, in any cases, if  $p \notin \ell(t_0, x_0)$  then  $p \notin \ell(t, x)$ . The proof is then complete.  $\square$

If we intensify slightly assumption (A2), then we have a stronger result as in the following theorem.

**Theorem 2.10.** Assume (A1) and (A2). In case that  $H(t, \cdot)$  is a concave function, we assume in addition that  $H(t, \cdot)$  is strictly concave for a.e.  $t$  in  $(0, T)$ . Let  $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ ,  $p_0 = \sigma_y(y) \in \ell(t_0, x_0)$  and let  $\mathcal{C} : x = x_0 + \int_{t_0}^t H_p(\tau, p_0) d\tau$ ,  $(t, x) \in \Omega$  be a characteristic curve of type (I) at  $(t_0, x_0)$ . Then  $\ell(t, x) = \{p_0\}$  for all  $(t, x) \in \mathcal{C}$ ,  $0 \leq t < t_0$ .



*Proof.* We use the notation as in the proof of Theorem 2.9. Let  $p \in \ell(t_1, x_1)$  where  $(t_1, x_1) \in \mathcal{C}$  and  $t_1 \in [0, t_0]$ . We consider two cases:

*Case 1.*  $\eta'(t, p) \leq 0$ ,  $t \in [0, t_0]$ . (The convexity condition of  $H(t, \cdot)$  will be also the case). Then  $0 = \eta(t_1, p) \leq \eta(0, p) \leq 0$ . Therefore  $\eta(0, p) = 0$  or  $\langle y, p - p_0 \rangle = \sigma^*(p) - \sigma^*(p_0)$ . Arguing as in the proof of Theorem 2.9, we have  $p = p_0$ .

*Case 2.*  $\eta'(t, p) \geq 0$ ,  $t \in [0, t_0]$ . (The concavity condition of  $H(t, \cdot)$  will be also the case). Then  $0 = \eta(t_1, p) \leq \eta(t_0, p) \leq 0$ . Therefore  $\eta(t_0, p) = 0$  and thus  $p \in \ell(t_0, x_0)$ . We will prove that  $p = p_0$ . Indeed, suppose contrarily that  $p \neq p_0$ .

If  $H(t, \cdot)$  is strictly concave for a.e.  $t \in (0, T)$ , then  $\eta'(t, p) > 0$  a.e. in  $(0, t_0)$ . Then  $0 = \eta(t_1, p) < \eta(t_0, p) \leq 0$ . This is a contradiction.

Now, assume that  $H(t, p) = g(t)h(p) + k(t)$ . Since  $p, p_0 \in \ell(t_0, x_0)$ , then it is obvious that  $\eta(t_0, p) = \varphi(t_0, x_0, p) - \varphi(t_0, x_0, p_0) = 0$ . This implies that

$$(2.15) \quad \langle x_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = \int_0^{t_0} (H(\tau, p) - H(\tau, p_0)) d\tau$$

Subtracting both sides by  $\langle \int_0^{t_0} H_p(\tau, p_0) d\tau, p - p_0 \rangle = (\int_0^{t_0} g(\tau) d\tau) \langle h_p(p_0), p - p_0 \rangle$ , and noticing that  $y = x_0 - \int_0^{t_0} H_p(\tau, p_0) d\tau$ , we get

$$(2.16) \quad \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = (\int_0^{t_0} g(\tau) d\tau) (h(p) - h(p_0) - \langle h_p(p_0), p - p_0 \rangle)$$

As mentioned before, since  $p_0 = \sigma_y(y)$ , then  $y \in \partial\sigma^*(p_0)$ . Arguing in case 1, we see that  $\langle y, p - p_0 \rangle < \sigma^*(p) - \sigma^*(p_0)$ , and since  $(\int_0^{t_0} g(\tau) d\tau) \neq 0$ , thus from (2.16) we deduce that

$$h(p) - h(p_0) - \langle h_p(p_0), p - p_0 \rangle \neq 0$$

and that

$$\eta'(t, p) = (\langle h_p(p_0), p - p_0 \rangle - (h(p) - h(p_0))) g(t)$$

does not change its sign on  $[0, t_0]$  since  $g(t)$  does not change its sign by assumption.

Using the strictly monotone property of function  $\eta(t, p)$  on  $[0, t_0]$  we see that,  $0 = \eta(t_1, p) < \eta(t_0, p) \leq 0$ . This also yields a contradiction. Thus  $p = p_0$  and consequently,  $\ell(t, x) = \{p_0\}$  for all  $(t, x) \in \mathcal{C}$ ,  $0 \leq t < t_0$ .  $\square$

We have seen that, if the characteristic curve  $\mathcal{C}$  is of type (I) at  $(t_0, x_0)$  then it is of the type (I) at any point  $(t, x) \in \mathcal{C}$ ,  $0 \leq t \leq t_0$ . Nevertheless, for the characteristic curve of type (II), we have the following:

**Theorem 2.11.** *Assume (A1). In addition, suppose that  $H, \sigma$  are of class  $C^2$ . Take  $(t_0, x_0) \in \Omega$  and let  $\mathcal{C} : x = x(t) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y_0)) d\tau$  be a characteristic curve of type (II) at  $(t_0, x_0)$ . Then there exists  $\theta \in (0, t_0)$  such that  $\mathcal{C}$  is of type (I) at  $(\theta, x(\theta))$  and  $\mathcal{C}$  is of type (II) for all point  $(t, x) \in \mathcal{C}$ ,  $t \in (\theta, t_0]$ .*

*Proof.* Let  $\mathcal{C} : x = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y_0)) d\tau$  be the characteristic curve of type (II) at  $(t_0, x_0)$  emanating from  $y_0$ . Then  $\sigma_y(y_0) \in \sigma_y(\ell^*(t_0, x_0)) \setminus \ell(t_0, x_0)$ .

By the Cauchy method of characteristics, the function defined by Hopf-type formula  $u(t, x)$  coincides with the local  $C^2$  solution of the Problem (2.1)-(2.2) see

[6]. Then there exists  $t_1 \in (0, t_0)$  such that  $u(t, x)$  is differentiable at any point  $(t, x(t)) \in \mathcal{C}$ ,  $u_x(t, x) = \sigma_y(y_0)$  and  $\ell(t, x) = \{\sigma_y(y_0)\}$ ,  $0 \leq t \leq t_1$ . Let

$$\theta = \sup\{t_1 \in [0, t_0] \mid \ell(s, x(s)) = \{\sigma_y(y_0)\}, 0 \leq s \leq t_1\}.$$

Since the multivalued mapping  $(t, x) \mapsto \ell(t, x)$  is upper semicontinuous, then we get that  $\sigma_y(y_0) \in \ell(\theta, x(\theta))$ . It is obvious that,  $\theta < t_0$  since  $\sigma_y(y_0) \notin \ell(t_0, x_0)$  and  $\mathcal{C}$  is of type (I) at  $(\theta, x(\theta))$ . On the other hand, for  $t \in (\theta, t_0]$ ,  $\mathcal{C}$  is of type (II) at  $(t, x(t))$  by the definition of  $\theta$ .  $\square$

For a locally Lipschitz function, it is promising to use the notion of sub- and superdifferential as well as reachable gradients, see [5], e.g., to study its differentiability. We use Theorem 2.10 to establish a relationship between  $\ell(t_0, x_0)$  and the set of reachable gradients. First, we briefly recall definitions of some kind of differentials as follows.

**Definition 2.12.** Let  $u = u(t, x) : \Omega \rightarrow \mathbb{R}$  and let  $(t_0, x_0) \in \Omega$ . For  $(h, k) \in \mathbb{R} \times \mathbb{R}^n$  we denote by

$$\tau(p, q, h, k) = \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - ph - \langle q, k \rangle}{\sqrt{|h|^2 + |k|^2}},$$

$$D^+u(t_0, x_0) = \{(p, q) \in \mathbb{R}^{n+1} \mid \limsup_{(h,k) \rightarrow (0,0)} \tau(p, q, h, k) \leq 0\}$$

$$D^-u(t_0, x_0) = \{(p, q) \in \mathbb{R}^{n+1} \mid \liminf_{(h,k) \rightarrow (0,0)} \tau(p, q, h, k) \geq 0\},$$

here  $p \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$ .

Then  $D^+u(t_0, x_0)$  (resp.  $D^-u(t_0, x_0)$ ) is called the *superdifferential* (resp. *subdifferential*) of  $u(t, x)$  at  $(t_0, x_0)$ .

We also define the set  $D^*u(t_0, x_0)$  of *reachable gradients* of  $u(t, x)$  at  $(t_0, x_0)$  as follows:

$\mathbb{R}^{n+1} \ni (p, q) \in D^*u(t_0, x_0)$  if and only if there exists a sequence  $(t_k, x_k)_k \subset \Omega \setminus \{(t_0, x_0)\}$  such that  $u(t, x)$  is differentiable at  $(t_k, x_k)$  and,

$$(t_k, x_k) \rightarrow (t_0, x_0), (u_t(t_k, x_k), u_x(t_k, x_k)) \rightarrow (p, q) \text{ as } k \rightarrow \infty.$$

If  $u(t, x)$  is a locally Lipschitz function, then  $D^*u(t, x) \neq \emptyset$  and it is a compact set ([5], p.54).

Now let  $u(t, x)$  be the Hopf-type formula and let  $(t_0, x_0) \in \Omega$ . We denote by

$$\mathcal{H}(t_0, x_0) = \{(-H(t_0, q), q) \mid q \in \ell(t_0, x_0)\}.$$

Then a relationship between  $D^*u(t_0, x_0)$  and the set  $\ell(t_0, x_0)$  is given by the following theorem.

**Theorem 2.13.** Assume (A1) and (A2). In case that  $H(t, \cdot)$  is a concave function, we assume in addition that  $H(t, \cdot)$  is strictly concave for a.e.  $t$  in  $(0, T)$ . Let  $u(t, x)$  be the Hopf-type formula for Problem (2.1)-(2.2). Then for all  $(t_0, x_0) \in \Omega$ , we have

$$D^*u(t_0, x_0) = \mathcal{H}(t_0, x_0).$$

*Proof.* Let  $(p_0, q_0)$  be an element of  $\mathcal{H}(t_0, x_0)$ , then  $p_0 = -H(t_0, q_0)$  for some  $q_0 \in \ell(t_0, x_0)$ . Let  $\mathcal{C}$  be the characteristic curve of type (I) at  $(t_0, x_0)$  defined as in Theorem 2.10. By assumption, all points  $(t, x) \in \mathcal{C}$ ,  $t \in [0, t_0]$  are regular. Put  $t_k = t_0 - 1/k$ , then  $\mathcal{C} \ni (t_k, x_k) \rightarrow (t_0, x_0)$  and  $(u_t(t_k, x_k), u_x(t_k, x_k)) = (-H(t_k, q_0), q_0) \rightarrow (-H(t_0, q_0), q_0) \in D^*u(t_0, x_0)$  as  $k \rightarrow \infty$ . Therefore,  $\mathcal{H}(t_0, x_0) \subset D^*u(t_0, x_0)$ .

On the other hand, let  $(p, q) \in D^*u(t_0, x_0)$  and  $(t_k, x_k)_k \subset \Omega \setminus \{(t_0, x_0)\}$  such that  $u(t, x)$  is differentiable at  $(t_k, x_k)$  and,  $(t_k, x_k) \rightarrow (t, x)$ ,  $(u_t(t_k, x_k), u_x(t_k, x_k)) \rightarrow (p, q)$  as  $k \rightarrow \infty$ . Since  $(u_t(t_k, x_k), u_x(t_k, x_k)) = (-H(t_k, q_k), q_k)$  for  $q_k \in \ell(t_k, x_k)$  and multivalued function  $\ell(t, x)$  is u.s.c, then letting  $k \rightarrow \infty$ , we see that  $q \in \ell(t_0, x_0)$  and  $p = \lim_{k \rightarrow \infty} -H(t_k, q_k) = -H(t_0, q)$ . Thus  $(p, q) \in \mathcal{H}(t_0, x_0)$ . The theorem is then proved.  $\square$

*Remark.* A general result for the correspondence between  $D^*u(t, x)$  and the set of minimizers of  $(CV)_{t,x}$  is established for convex Hamiltonian  $H(t, x, p)$  in  $p$  in [5], Th. 6.4.9, p.167.

### 3. REGULARITY OF HOPF-TYPE FORMULA

In this section we will study the sets of the form  $V = (0, t_*) \times \mathbb{R}^n \subset \Omega$  such that  $u(t, x)$  is continuously differentiable on them. Next, we show that if  $(t_0, x_0)$  is singular, then there exists another singular point  $(t, x)$  for  $t > t_0$  and  $x$  is near to  $x_0$ .

**Theorem 3.1.** *Assume (A1), (A2). Let  $u(t, x)$  be the viscosity solution of Problem (2.1) - (2.2) defined by Hopf-type formula (2.3). Suppose that there exists  $t_* \in (0, T)$  such that the mapping:  $y \mapsto x(t_*, y) = y + \int_0^{t_*} H_p(\tau, \sigma_y(y)) d\tau$  is injective. Then  $u(t, x)$  is continuously differentiable in the open strip  $(0, t_*) \times \mathbb{R}^n$ .*

*Proof.* Let  $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$  and let  $\mathcal{C}$  :

$$x = x_0 + \int_{t_0}^t H_p(\tau, p_0) d\tau$$

where  $p_0 = \sigma_y(y_0) \in \ell(t_0, x_0)$  be the characteristic curve going through  $(t_0, x_0)$  defined as in Proposition 2.5.

Let  $(t_*, x_*)$  be the intersection point of  $\mathcal{C}$  and plane  $\Delta^{t_*} : t = t_*$ . By assumption, the mapping  $y \mapsto x(t_*, y)$  is injective and  $\ell(t_*, x_*) \neq \emptyset$ , so there is unique a characteristic curve passing  $(t_*, x_*)$ . This characteristic curve is exactly  $\mathcal{C}$ . Therefore, we can rewrite  $\mathcal{C}$  as follows:

$$x = x_* + \int_{t_*}^t H_p(\tau, p_*) d\tau$$

where  $p_* \in \ell(t_*, x_*)$ .

Since  $\ell^*(t_*, x_*)$  is a singleton, so is  $\ell(t_*, x_*)$ . Consequently,  $\mathcal{C}$  is of type (I) at  $(t_*, x_*)$  and  $\ell(t, x) = \{p^*\}$  for all  $(t, x) \in \mathcal{C}$ , particularly at  $(t_0, x_0)$  and then,  $p^* = p_0$ . Applying Theorem 2.1 we see that  $u(t, x)$  is of class  $C^1$  in  $(0, t_*) \times \mathbb{R}^n$ .  $\square$

Note that at some point  $(t_0, x_0) \in \Omega$  where  $u(t, x)$  is differentiable there may be more than one characteristic curve goes through, that is  $\ell^*(t_0, x_0)$  may not be a singleton. Next, we have:

**Theorem 3.2.** *Assume (A1) and (A2). Moreover, let  $\sigma$  be Lipschitz on  $\mathbb{R}^n$ . Suppose that  $\ell(t_*, x)$  is a singleton for every point of the plane  $\Delta^{t_*} = \{(t_*, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$ ,  $0 < t_* \leq T$ . Then the function  $u(t, x)$  defined by Hopf-type formula (2.3) is continuously differentiable in the open strip  $(0, t_*) \times \mathbb{R}^n$ .*

*Proof.* Let  $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$ . Since  $\sigma(x)$  is convex and Lipschitz on  $\mathbb{R}^n$  then  $\text{dom } \sigma^* = \{q \in \mathbb{R} \mid \sigma^*(q) < +\infty\} = D$  is a bounded (and convex) in  $\mathbb{R}^n$ . We thus have  $\ell(t, x) \subset D$  for all  $(t, x) \in \Omega$ .

For each  $y \in \mathbb{R}^n$ , we put

$$\Lambda(y) = x_0 - \int_{t_*}^{t_0} H_p(\tau, p(y)) d\tau,$$

where  $p(y) \in \ell(t_*, y) \in D$ . Since the multi-valued function  $y \mapsto \ell(t_*, y)$  is u.s.c, see [8] and, by assumption  $\ell(t_*, y) = \{p(y)\}$  is a singleton for all  $y \in \mathbb{R}^n$ , then  $y \mapsto p(y)$  is continuous. Therefore the function  $\mathbb{R}^n \ni y \mapsto \Lambda(y) = x_0 - \int_{t_*}^{t_0} H_p(\tau, p(y)) d\tau$  is also continuous on  $\mathbb{R}^n$ .

Since  $p(y)$  is in the bounded set  $D$  and  $H_p(t, p)$  is continuous, there exists  $M > 0$  such that

$$|\Lambda(y) - x_0| \leq \int_{t_0}^{t_*} |H_p(\tau, p(y))| d\tau \leq M.$$

Therefore  $\Lambda$  is a continuous function from the closed ball  $B'(x_0, M)$  into itself. By Brouwer theorem,  $\Lambda$  has a fixed point  $x_* \in \mathbb{R}^n$ , i.e.,  $\Lambda(x_*) = x_*$ , hence

$$x_0 = x_* + \int_{t_*}^{t_0} H_p(\tau, p(x_*)) d\tau.$$

In other words, there exists a characteristic curve  $C$  of the type (I) at  $(t_*, x_*)$  described as in Theorem 2.9 passing  $(t_0, x_0)$ . Since  $\ell(t_*, x_*)$  is a singleton, so is  $\ell(t_0, x_0)$ . Applying Theorem 2.1, we see that  $u(t, x)$  is continuously differentiable in  $(0, t_*) \times \mathbb{R}^n$ .  $\square$

We note that the hypotheses of above theorems are equivalent to the fact that, there is unique characteristic curve of type (I) at points  $(t_*, x)$ ,  $x \in \mathbb{R}^n$  going through  $(t_0, x_0)$ . In general, at some point  $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$  where  $u(t, x)$  is differentiable there may be more than one characteristic curves of type (I) or (II) at points  $(t_*, x)$ ,  $x \in \mathbb{R}^n$ , that is  $\ell^*(t_*, x)$  may not be a singleton. Even neither is  $\ell(t_*, x)$ . Nevertheless, we have:

**Theorem 3.3.** *Assume (A1) and (A2). In case that  $H(t, \cdot)$  is a concave function, we assume in addition that  $H(t, \cdot)$  is strictly concave for a.e.  $t$  in  $(0, T)$ . Let  $u(t, x)$  be the viscosity solution of Problem (2.1) - (2.2) defined by Hopf-type formula. Suppose that there exists  $t_* \in (0, T)$  such that all characteristic curves passing  $(t_*, x)$ ,  $x \in \mathbb{R}^n$  are of type (I). Then  $u(t, x)$  is continuously differentiable in the open strip  $(0, t_*) \times \mathbb{R}^n$ .*

*Proof.* We argue similarly to the proof of Theorem 3.1. Let  $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$  and let  $\mathcal{C}$  :

$$x = x_0 + \int_{t_0}^t H_p(\tau, p_0) d\tau$$

where  $p_0 = \sigma_y(y_0) \in \ell(t_0, x_0)$  be the characteristic curve going through  $(t_0, x_0)$  defined as in Proposition 2.5.

Let  $(t_*, x_*)$  be the intersection point of  $\mathcal{C}$  and plane  $\Delta^{t_*} : t = t_*$ . Then we have

$$x_* = x_0 + \int_{t_0}^{t_*} H_p(\tau, p_0) d\tau$$

Therefore, we can rewrite  $\mathcal{C}$  as

$$x = x_* - \int_{t_0}^{t_*} H_p(\tau, p_0) d\tau + \int_{t_0}^t H_p(\tau, p_0) d\tau = x_* + \int_{t_*}^t H_p(\tau, p_0) d\tau$$

is also a characteristic curve passing  $(t_*, x_*)$ . By assumption,  $\mathcal{C}$  is of type (I) at this point, so all  $(t, x) \in \mathcal{C}$ ,  $0 \leq t < t_*$  are regular by Theorem 2.10. Thus,  $\ell(t_0, x_0)$  is a singleton. As before, we come to the conclusion of the theorem.  $\square$

Next, we study the propagation of singularities of viscosity solution of the Cauchy problem (2.1)-(2.2).

**Theorem 3.4.** *Assume (A1) and (A2). Let  $(t_0, x_0)$  be a singular point of  $u(t, x)$ . Then for each  $\epsilon > 0$  there exists  $\delta > 0$  such that for any  $t_* > t_0$ ,  $|t_* - t_0| \leq \delta$ , there exists  $x_* \in B'(x_0, \epsilon)$  such that  $(t_*, x_*)$  is also a singular point.*

*Proof.* We prove by contradiction. Let  $\epsilon > 0$ . By assumption (A1), for  $(t, x) \in \Omega$  such that  $x \in B'(x_0, \epsilon)$  and  $|t - t_0| \leq \alpha$ , ( $\alpha$  is small enough), there exist numbers  $r_x$  and  $N_x$  such that  $|t' - t| + |x' - x| < r_x$  then  $\ell(t', x') \subset B'(0, N_x)$ . Therefore, we can cover the closed ball  $B'(x_0, \epsilon)$  by a finite number balls centered at  $x_i$  with radii  $r_{x_i}/2$ ,  $i = 1, \dots, k$ . Then we can find positive numbers  $m = \min\{r_{x_i}/2, \alpha\}$ ,  $M = \max\{N_{x_i}\}$ ,  $i = 1, \dots, k$  such that for all  $(t, x) \in \Omega$ ,  $|t - t_0| \leq m$ ,  $|x - x_0| \leq \epsilon$  then  $\ell(t, x) \subset B'(0, M)$ . Now we choose  $\delta \in (0, m]$  satisfying  $\delta \sup_{|t-t_0| \leq m, |p| \leq M} |H_p(t, p)| \leq \epsilon$  and take  $t_* > t_0$  such that  $t_* - t_0 \leq \delta$ .

If every point  $(t_*, y)$  where  $y \in B'(x_0, \epsilon)$  is regular, then  $\ell(t_*, y) = \{p(y)\} = \{p(t_*, y)\}$  is a singleton. Since the multi-valued function  $y \mapsto \ell(t_*, y)$  is u.s.c, then  $y \mapsto p(y)$  is continuous on  $B'(x_0, \epsilon)$ . Therefore the function  $\mathbb{R}^n \ni y \mapsto \Lambda(y) = x_0 - \int_{t_*}^t H_p(\tau, p(y)) d\tau$  is also continuous.

Note that, if  $y \in B'(x_0, \epsilon)$  then

$$|\Lambda(y) - x| \leq \int_{t_0}^{t_*} |H_p(\tau, p(y))| d\tau \leq \delta \sup_{|t-t_0| \leq m, |p| \leq M} |H_p(t, p)| \leq \epsilon.$$

Therefore  $\Lambda$  is a continuous function from the closed ball  $B'(x_0, \epsilon)$  into itself. By Brouwer theorem,  $\Lambda$  has a fixed point  $x_* \in B'(x_0, \epsilon)$ , i.e.,  $\Lambda(x_*) = x_*$ , hence,

$$x_0 = x_* + \int_{t_*}^{t_0} H_p(\tau, p(x_*)) d\tau.$$

In other words, there exists a characteristic curve  $\mathcal{C}$  of the type (I) at  $(t_*, x_*)$  described as in Theorem 2.9 passing  $(t_0, x_0)$ . Since  $\ell(t_*, x_*)$  is a singleton, so is  $\ell(t_0, x_0)$ . This contradicts to the hypothesis.  $\square$

*Remark 3.5.* If  $(t_0, x_0) \in \Omega$  is a singular point for  $u(t, x)$  and  $\epsilon > 0$ , then there exists  $\delta_1 = \delta > 0$  such that for any  $t \in [t_0, t_0 + \delta_0]$  we can pick out  $x = x(t) \in B'(x_0, \epsilon)$  so that  $(t, x)$  is singular. Put  $x_1 = x(t_1)$  where  $t_1 = t_0 + \delta_1$ . By induction, we can find  $(\delta_k)_k$  and  $x_k = x(t_k)$ ,  $t_k = t_{k-1} + \delta_k$  so that  $(t_k, x_k)$  is singular. Since  $\delta_k > 0$  is dependent on  $(t_k, x_k)$  there are two possibilities:

$$\sum_{k=1}^{\infty} \delta_k < T \quad \text{or} \quad \sum_{k=1}^{\infty} \delta_k \geq T.$$

In the first case, the singularity of  $u(t, x)$  constructed by this way may not propagates to the boundary, otherwise the singularity of  $u(t, x)$  approaches to boundary  $t = T$  in the second case. Nevertheless, if we assume  $\sigma(x)$  is Lipschitz on  $\mathbb{R}^n$  as an additional condition, then the number  $\delta > 0$  can be chosen independently of  $(t_i, x_i)$ ,  $i = 1, 2, \dots$

We have the following:

**Theorem 3.6.** *Assume (A1), (A2). Moreover, let  $\sigma(x)$  be a Lipschitz function on  $\mathbb{R}^n$ . For each  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $(t_0, x_0)$  is a singular point for  $u(t, x)$ , then for any  $t_1 \in [t_0, t_0 + \delta]$  there exists  $x_1 \in B'(x_0, \epsilon)$  such that  $(t_1, x_1)$  is also a singular point.*

*Proof.* Since  $\sigma(x)$  is convex and Lipschitz, then  $D = \text{dom}\sigma^*$  is bounded. Hence,  $D \subset B'(0, M)$  for some positive number  $M$ . We choose a fixed number  $\delta > 0$  such that  $\delta \sup_{0 \leq t \leq T, |p| \leq M} |H_p(t, p)| \leq \epsilon$ .

We argue as in the proof of Theorem 3.4 Let  $(t_0, x_0)$  be a singular point for  $u(t, x)$ . If there is  $t_* \in (t_0, t_0 + \delta]$  such that  $(t_*, y)$  is regular for all  $y \in B'(x_0, \epsilon)$  then the mapping

$$y \mapsto \Lambda(y) = x_0 - \int_{t_*}^t H_p(\tau, p(y)) d\tau$$

is continuous from  $B'(x_0, \epsilon)$  into itself. Thus, the mapping has a fixed point  $x_* \in B'(x_0, \epsilon)$ . This implies that there is a characteristics  $C$  of type (I) at  $(t_*, x_*)$  passing  $(t_0, x_0)$  and so  $(t_0, x_0)$  is regular. This is a contradiction.  $\square$

**Corollary 3.7.** *Assume (A1), (A2) and  $\sigma(x)$  is Lipschitz on  $\mathbb{R}^n$ . If  $u(t, x)$  has a singular point  $(t_0, x_0) \in \Omega$ , then for any  $\epsilon > 0$  and  $t > t_0$ , we can find another singular point  $(t, x)$  such that  $|x - x_0| \leq m\epsilon$ , for some  $m \in \mathbb{N}$ . Therefore the singular points of  $u(t, x)$  propagate with respect to  $t$  as  $t$  tends to  $T$ .*

*Proof.* Arguing as in Remark 3.5, we see that for  $\epsilon > 0$  and  $t > t_0$ ,  $|t - t_0| < \delta$ , after several steps, there is  $m \in \mathbb{N}$  such that  $m\delta < t \leq (m+1)\delta$ . Then there exists  $x_m \in B'(x_{m-1}, \epsilon)$  such that  $(t, x_m)$  is singular and  $|x_m - x_0| \leq |x_m - x_{m-1}| + \dots + |x_1 - x_0| \leq m\epsilon$ .  $\square$

**Example.** We consider the following problem

$$\frac{\partial u}{\partial t} - 2t \ln(1 + u_x^2) = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(0, x) = \frac{x^2}{2}, \quad x \in \mathbb{R}.$$

A viscosity solution defined by Hopf-type formula of this problem is:

$$u(t, x) = \max_{y \in \mathbb{R}} \left\{ xy - \frac{y^2}{2} + t^2 \ln(1 + y^2) \right\}$$

Let  $\varphi(t, x, y) = xy - \frac{y^2}{2} + t^2 \ln(1 + y^2)$ , then  $\varphi_y(t, x, y) = x - y + \frac{2t^2 y}{1+y^2}$ .

A simple computation shows that at point  $(t_0, x_0) = (\sqrt{2}, \frac{2}{5})$ , we have  $\varphi_y(\sqrt{2}, \frac{2}{5}, y) = 0 \Leftrightarrow y_1 = 2; y_2 = \frac{-4+\sqrt{11}}{5}, y_3 = \frac{-4-\sqrt{11}}{5}$  and the function  $\varphi(t_0, x_0, y)$  attains its maximum at  $y_1 = 2$ .

There are three characteristic curves that go through the point  $(\sqrt{2}, \frac{2}{5})$  as follows:

$C_1 : x = 2 - \frac{4t^2}{5}$ , starting at  $y=2$  and

$C_i = y_i - \frac{2y_i t^2}{1+y_i^2}$ ,  $i = 2, 3$ , starting at  $y_2 = \frac{-4+\sqrt{11}}{5}$ ,  $y_3 = \frac{-4-\sqrt{11}}{5}$ .

We see that  $C_1$  is the characteristic curve of type (I) at  $(t_0, x_0)$  and  $C_2, C_3$  are the characteristic curves of type (II) at  $(\sqrt{2}, \frac{2}{5})$  since  $\ell(\sqrt{2}, \frac{2}{5}) = \{\sigma'(y_1)\} = \{2\}$  and  $\sigma_y(y_i) \notin \ell(\sqrt{2}, \frac{2}{5})$ ,  $i = 2, 3$ . Note that,  $(\sqrt{2}, \frac{2}{5})$  is a regular point of  $u(t, x)$ .

Now let  $(t_1, x_1) = (t_1, 0)$  and let the characteristics  $\mathcal{C}$  starting  $y \in \mathbb{R}$  go through  $(t_1, 0)$ . Then  $y$  is a root of equation  $y - \frac{2t_1^2 y}{1+y^2} = 0$ .

If  $0 \leq t_1 \leq \frac{1}{\sqrt{2}}$  then  $(t_1, 0)$  is regular point of  $u(t, x)$  and  $\mathcal{C}_1 : x = 0$  is of type (I) at  $(t_1, 0)$ .

If  $t_1 > \frac{1}{\sqrt{2}}$  then  $(t_1, 0)$  is singular, since  $\ell(t_1, 0) = \{y_2, y_3\}$ , where  $y_2 = \sqrt{2t_1^2 - 1}$ ,  $y_3 = -\sqrt{2t_1^2 - 1}$ . In this case, the characteristic curves  $\mathcal{C}_2$  and  $\mathcal{C}_3$  starting at  $y_2$  and  $y_3$  are of type (I), and  $\mathcal{C}_1$  is of type (II) at  $(t_1, 0)$ .

Let  $t_* = \frac{1}{\sqrt{2}}$ . We have  $\varphi(\frac{1}{\sqrt{2}}, x, y) = xy - \frac{y^2}{2} + \frac{1}{2} \ln(1 + y^2)$ , then  $\varphi'_y(\frac{1}{\sqrt{2}}, x, y) = x - y + \frac{y}{1+y^2}$  and  $\varphi''_y(\frac{1}{\sqrt{2}}, x, y) = -y^2 \frac{3+y^2}{(1+y^2)^2} < 0$ ,  $y \neq 0$ . Therefore  $\ell(\frac{1}{\sqrt{2}}, x)$  is a singleton for all  $x \in \mathbb{R}$ . Applying Theorem 3.2, we see that the solution  $u(t, x)$  is continuously differentiable on the strip  $(0, \frac{1}{\sqrt{2}}) \times \mathbb{R}^n$ .

At last, the segment  $x = 0; t \in (\frac{1}{\sqrt{2}}, T]$  is a set of singular points for  $u(t, x)$ . So the singularities of  $u(t, x)$  propagate to the boundary.

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